Length of a curve. Green Theorem. Path independence and the existence of a primitive.

Sunday, September 24, 2023 9:05 AM

Length and integral with respect to length

Real notation:	Complex notation.
$\ell(\gamma) := \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$	$\ell(\gamma) := \int_a^b \lvert z'(t)\rvert\, dt$
$\int_\gamma f\, ds := \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$	$\int_\gamma f(z) \lvert dz\rvert := \int_a^b f(z(t)) \lvert z'(t)\rvert\, dt$

Independent of parameterization and orientation.

Lemma $\left| \oint_\gamma f(z)\, dz \right| \le \int_\gamma |f(z)|\,|dz| \le \sup_{z\in\gamma} |f|\, \ell(\gamma)$.

Proof $\left| \oint f(z)\, dz \right| = \left| \int_a^b f(z(t))\, z'(t)\, dt \right| \le \int_a^b |f(z(t))|\,|z'(t)|\, dt \ (=)$

$(=) \int_\gamma |f(z)|\,|dz|$ and $(\le) \sup_{z\in\gamma} |f| \int_a^b |z'(t)|\, dt = \sup_{z\in\gamma} |f|\, \ell(\gamma)$ ∎

Let γ be an arc.

Lemma. Let $f_n, f : \gamma \to \mathbb{C}$ - continuous, $f_n \rightrightarrows f$ uniformly on γ.

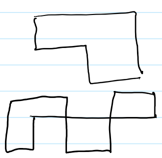
Then $\oint_\gamma f_n(z)\, dz \to \oint_\gamma f(z)\, dz$

Proof. $\left| \oint_\gamma f_n(z)\, dz - \oint_\gamma f(z)\, dz \right| = \left| \oint_\gamma (f_n(z) - f(z))\, dz \right| \le \sup_\gamma |f_n(z) - f(z)|\, \ell(\gamma) \to 0$ ∎

Corollary. Let $f_n, f : \gamma \to \mathbb{C}$, continuous, $\sum_{n=1}^\infty f_n = f$ uniformly on γ. Then $\oint_\gamma f(z)\, dz = \sum_{n=1}^\infty \oint_\gamma f_n(z)\, dz$

Notations: γ is

1) simple, if $\gamma : [a,b] \to \mathbb{C}$ - injective
2) closed, if $\gamma(a) = \gamma(b)$
3) closed simple, if $\gamma(a) = \gamma(b)$, $\gamma : [a,b) \to \mathbb{C}$ injective
4) polygonal arc, if $\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n$, where each γ_k is parallel to one of the axes.



Def. (continuous) Vector field $\begin{pmatrix} p(z) \\ q(z) \end{pmatrix}$ (differential form $p\,dx + q\,dy$) is exact or path-independent in a region D if for any two arcs $\gamma, \gamma' \subset D$, with the same start point and the same endpoint we have $\oint_\gamma p(z)\,dx + q(z)\,dy = \oint_{\gamma'} p(z)\,dx + q(z)\,dy$

Theorem. TFAE:

1) $\begin{pmatrix} p(z) \\ q(z) \end{pmatrix}$ is exact.

2) $\forall \gamma \subset D$, closed : $\oint_\gamma p\,dx + q\,dy = 0$

3) $\forall \gamma \subset D$, polygonal, closed : $\oint_\gamma p\,dx + q\,dy = 0$

4) $\exists\, V : D \to \mathbb{C}$, real-differentiable,

$$\frac{\partial V}{\partial x} = p, \quad \frac{\partial V}{\partial y} = q. \quad \left(\nabla V = (p, q); \ dV = p\,dx + q\,dy \right)$$

V is called <u>potential</u> of $\begin{pmatrix} p \\ q \end{pmatrix}$. $\begin{pmatrix} p \\ q \end{pmatrix}$ is called <u>gradient field</u> of V.

<u>Proof</u>. 1) $\Rightarrow$ 2) - Consider $\gamma + \gamma$, then

$$\oint_\gamma \underset{\text{indep.}}{\overset{\text{path-}}{=}} \oint_{\gamma+\gamma} \underset{\text{additivity}}{=} 2\oint_\gamma \Rightarrow \oint_\gamma = 0$$

2) $\Rightarrow$ 1) Consider the closed arc $\gamma - \gamma'$.

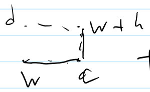
$$\int_{\gamma - \gamma'} = 0 \Rightarrow \int_\gamma - \int_{\gamma'} = 0.$$

2) $\Rightarrow$ 3) - obvious.

3) $\Rightarrow$ 4) Fix $w_0 \in D$. For $w \in D$, define

$V(w) := \oint_\gamma P(z)\,dx + Q(z)\,dy$ for any polygonal path γ from w_0 to w. Does not depend on γ, by 3).

Let $B(w, r) \in D$, take $h = c + id$, $|h| < r$.



Then let $\gamma_h = [w, w+c] \cup [w+c, w+h]$ - polygonal.

If $V(w) = \oint_\gamma p(z)\,dx + q(z)\,dy$ then

$V(w+h) = \oint_{\gamma + \gamma_h} \ldots$

$$V(w+h) - V(w) = \int_{\gamma_h} p(z)\,dx + q(z)\,dy = \int_0^c p(w+x)\,dx + \int_0^d q(w+c+iy)\,dy$$

So $$\left| V(w+h) - V(w) - p(w)c - q(w)d \right| = \left| \int_0^c p(w+x) - p(w)\,dx + \int_0^d q(w+c+iy) - q(w)\,dy \right| \le c \sup_{w' \in B(w,r)} |P(w') - P(w)| + d \sup_{w' \in B(w,r)} |q(w') - q(w)|$$

As $r \to 0$, the bound is $o(|h|)$.

So V is real-differentiable, $\frac{\partial V}{\partial x} = p$, $\frac{\partial V}{\partial y} = q$.

4) $\Rightarrow$ 1) $\oint_\gamma p\,dx + q\,dy = \oint_\gamma \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy = \int_a^b \left(\frac{\partial V}{\partial x} x'(t) + \frac{\partial V}{\partial y} y'(t) \right) dt =$

$V(\gamma(b)) - V(\gamma(a))$ ∎

For complex differential:

$f(z)\,dz = f(z)\,dx + i\,f(z)\,dy$.

It is exact if and only if $\exists F$:

$\frac{\partial F}{\partial x} = f(z)$, $\frac{\partial F}{\partial y} = i f(z)$. I.e. $\frac{\partial F}{\partial z} = \frac{1}{2}\left(f(z) - i\cdot i f(z)\right) = f(z)$.

$\frac{\partial F}{\partial \bar z} = \frac{1}{2}\left(f(z) + i\cdot i f(z)\right) = 0$.

So F is complex-differentiable and $F'(z) = f(z)$.

Theorem. $f(z)\,dz$ is exact iff $\exists$ analytic F such that $F' = f$. F is called antiderivative.

$\forall \gamma \subset D$ from z_1 to z_2, $\oint_\gamma f(z)\,dz = F(z_2) - F(z_1)$.

Remark (important) Not all analytic functions are exact!

$f(z) = \frac{1}{z}$ in $D = \mathbb{C}\setminus\{0\}$. $\oint_\gamma \frac{dz}{z} = 2\pi i \neq 0$.

But locally all analytic functions have antiderivative.

Theorem (local behavior). Continuous vector field is exact in $D = B(z_0, r)$ iff for any rectangle $R \subset B(z_0, r)$,

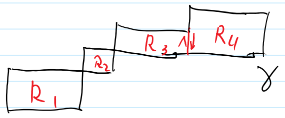
$\oint_{\partial R} p\,dx + q\,dy = 0$

Proof. Need to proof only "if" ($\Leftarrow$) part.

Consider polygonal closed curve γ in D. Let $D' \subset D$ be the interior of γ, $\partial D' = \gamma$.

Cut it interior into rectangles $R_1 \cup R_2 \ldots \cup R_n = D'$. ($D'$ is open, but not always a region)

Note that $\oint_\gamma p\,dx + q\,dy = \sum_{j=1}^{n} \oint_{\partial R_j} p\,dx + q\,dy = 0$



The same holds for $D = B(z_0, r) \setminus \{z_1, \ldots, z_n\}$ - finite collection of points.

Need to consider any rectangle R: $\partial R \subset D$.

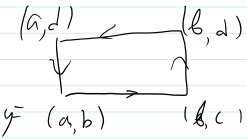
Assume that $f \in \mathcal{A}(B(z_0, r))$ and f' is continuous. Then

Assume that $f \in \mathcal{A}(B(z_0,r))$ and f' is continuous. Then we can use Green theorem to show that f has antiderivative.

Thm (Green). Let D be a domain, ∂D - piecewise differentiable curve. Orient ∂D counterclockwise. Let $\binom{p}{q}$ - continuous, and both p and q continuously differentiable in $D \cup \partial D$. Then

$$\oint_{\partial D} p\,dx + q\,dy = \iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy.$$

For $D = R = [a,b] \times [c,d]$, it is very easy:



$$\int_a^b p(x,c)\,dx - \int_a^b p(x,d)\,dx + \int_c^d q(b,y)\,dy - \int_c^d q(a,y)\,dy =$$

$$\int_c^d \big(q(b,y) - q(a,y)\big)\,dy - \int_a^b \big(p(x,d) - p(x,c)\big)\,dx \underset{FTC}{=} \int_a^b \int_c^d \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy.$$

Complex form: If f is continuously real-differentiable in $D \cup \partial D$, then

$$\oint_{\partial D} f(z)\,dz = \oint_{\partial D} f\,dx + if\,dy = \iint_D \left(\frac{\partial (if)}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy =$$

$$i \iint \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy = \boxed{2i \iint \frac{\partial f}{\partial \bar z}\,dx\,dy}$$

Corollary. If $f \in \mathcal{A}(B(z_0,r))$ and f' is continuous, then $\exists F \in \mathcal{A}(B(z_0,r)) : F' = f$.

Proof. We need to prove that for any rectangle $R \subset B(z_0,r)$, $\oint_{\partial R} f(z)\,dz = 0$. But by Green Theorem, $\oint_{\partial R} f(z)\,dz = 2i \iint_R \frac{\partial f}{\partial \bar z}\,dx\,dy = 0$ ∎

Classical Cauchy Theorem:

If $f \in \mathcal{A}(B(z_0,r))$ and f' is continuous, then $\forall \gamma \subset B(z_0,r)$, closed $\oint_\gamma f(z)\,dz = 0$.